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Necessary Conditions for Normed Convergence of Critical Multitype Bienaymé–Galton–Watson Processes without Variance

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The condition on the offspring distribution in the critical multitype Bienaymé–Galton–Watson process without variance, which was previously shown to be sufficient for the existence of the analogue of the exponential limit law, is now shown also to be necessary. This completely extends previous one-type work of R. S. Slack.

1. INTRODUCTION

Suppose that $\{\mathbf{Z}_n \equiv (Z_n^{(1)},..., Z_n^{(d)})\}$ is a d-type positively regular, nonsingular, and critical Bienaymé–Galton–Watson process. In [2] it was shown that if a certain regularity condition is imposed on the offspring p.g.f. $\mathbf{F}(\mathbf{s}) \equiv (F^{(1)}(\mathbf{s}),..., F^{(d)}(\mathbf{s}))$ then there exists a positive normalizing sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} \Pr[a_n\mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}] = H(\mathbf{s}) \tag{1.1}$$

exists where H is a proper and nondegenerate distribution function. This condition was best phrased in terms of the mean value expansion of Joffe and Spitzer [4]

$$\mathbf{1} - \mathbf{F}(\mathbf{s}) = (M - E(\mathbf{s}))(\mathbf{1} - \mathbf{s}), \tag{1.2}$$

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M being the offspring expectation matrix. Let $\mathbf{v}$, $\mathbf{u}$ denote its left and right eigenvectors, respectively, corresponding to eigenvalue unity, normalized so that $\mathbf{v} \cdot \mathbf{u} = 1 = \mathbf{1} \cdot \mathbf{u}$. Define for positive and small scalar x

$$\Lambda(x) = \mathbf{v}E(\mathbf{1} - x\mathbf{u})\mathbf{u}.$$

Then the appropriate condition is given by

$$\Lambda(x) = x^{\alpha}L(x) \tag{1.3}$$

for some $0 < \alpha \leqslant 1$ and a function L slowly varying at 0. Furthermore, the constants may be taken as $a_n = \mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(\mathbf{0}))$, $\mathbf{F}_n$ denoting the nth functional iterate of $\mathbf{F}$. This result extended one-dimensional work of Slack [7] whose condition

$$F(s) = s + (1 - s)^{1+\alpha}L(1 - s) \tag{1.4}$$

was taken as the basis for (1.3). In a sequel [8] he showed that (1.4) was also necessary for convergence with the above constants a_n. The purpose of the present article is to show that (1.3) is the correct generalization to the multitype case by proving its necessity if (1.1) is to hold. The proof is along the same lines as [8] with appropriate modifications for operating in higher dimensions.

THEOREM. *Let* $\mathbf{e}_i$, $1 \leqslant i \leqslant d$, *be a unit vector consisting of zeros except for a* 1 *in position* i. *Suppose for some* i, *there is a vector* $\mathbf{w}$ *with* $\mathbf{w} \cdot \mathbf{v} > 0$ *and a nondegenerate* (*but possibly defective*) *distribution function* $G(x; \mathbf{w}, i)$ *at all of whose continuity points* x,

$$\lim_{n\to\infty} \Pr[a_n\mathbf{Z}_n \cdot \mathbf{w} \leqslant x \mid \mathbf{Z}_n \neq \mathbf{0}, \mathbf{Z}_0 = \mathbf{e}_i] = G(x; \mathbf{w}, i). \tag{1.5}$$

Then:

(a) *Equation* (1.3) *must hold.*

(b) *There exists a proper and nondegenerate distribution function* $H(\mathbf{s})$ *such that for any nonnegative integral lattice vector* $\mathbf{k} \neq \mathbf{0}$

$$\lim_{n\to\infty} \Pr[a_n\mathbf{Z}_n \leqslant \mathbf{s} \mid \mathbf{Z}_n \neq \mathbf{0}, \mathbf{Z}_0 = \mathbf{k}] = H(\mathbf{s})$$

for all $\mathbf{0} \leqslant \mathbf{s}$ *independent of* $\mathbf{k}$.

2. Preparatory Lemmas

For the sequel define for positive t, a_t to be $a_{[t]}$ where $[t]$ is the greatest integer in t.

LEMMA 0.

$$\lim_{n\to\infty} \frac{\mathbf{v}\cdot(\mathbf{F}(\mathbf{1}-a_n\mathbf{u})-\mathbf{F}_n(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{F}_{n+1}(\mathbf{0})-\mathbf{F}_n(\mathbf{0}))} = 1. \tag{2.1}$$

Proof. By the mean value theorem $\mathbf{v}\cdot(\mathbf{F}(\mathbf{x})-\mathbf{F}(\mathbf{z}))/\mathbf{v}\cdot(\mathbf{x}-\mathbf{z}) \to 1$ as $\mathbf{x}$ and $\mathbf{z}\to\mathbf{1}$. Since the nth term of the limiting sequence in (2.1) may be decomposed as

$$\frac{\mathbf{v}\cdot(\mathbf{F}(\mathbf{1}-a_n\mathbf{u})-\mathbf{F}_n(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{1}-a_n\mathbf{u}-\mathbf{F}_{n-1}(\mathbf{0}))} \cdot \frac{\mathbf{v}\cdot(\mathbf{F}_n(\mathbf{0})-\mathbf{F}_{n-1}(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{F}_{n+1}(\mathbf{0})-\mathbf{F}_n(\mathbf{0}))}$$

and both factors therefore tend to 1, the lemma obtains. ▮

LEMMA 1.

$$\liminf_{n\to\infty} n \frac{\mathbf{v}\cdot(\mathbf{F}_{n+1}(\mathbf{0})-\mathbf{F}_n(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))} \geqslant 1. \tag{2.2}$$

Proof. For $0 \leqslant x \leqslant 1$ define

$$f(x) = \mathbf{v}\cdot\mathbf{F}(\mathbf{1}-(1-x)\mathbf{u}) \equiv \mathbf{v}\cdot\mathbf{F}(\mathbf{1}-\mathbf{u}+x\mathbf{u}).$$

(Note that f is not a p.g.f. since $f(1) = \mathbf{v}\cdot\mathbf{1} > 1$.) $f'(x) = \mathbf{v}M(\mathbf{1}-\mathbf{u}+x\mathbf{u})\mathbf{u}$ where $M(\cdot)$ is the differential map of $\mathbf{F}(\cdot)$. Each component of M being a power series in x with nonnegative coefficients it follows that $1-f'(x)$ is concave. Applying this fact with a geometric argument to the identity

$$f(x) - x + 1 - \mathbf{v}\cdot\mathbf{1} = \int_x^1 (1-f'(t))\,dt$$

leads directly to the inequalities

$$\tfrac{1}{2}(1-x)(1-f'(x)) \leqslant f(x) - x + 1 - \mathbf{v}\cdot\mathbf{1} \leqslant (1-x)(1-f'(x)).$$

Define $h(x) = (1-x)/(f(x) - x + 1 - \mathbf{v}\cdot\mathbf{1})$. Substitution shows that $(1-x)\,h'(x)/h(x) = -1 + (1-x)(1-f'(x))/(f(x)-x+1-\mathbf{v}\cdot\mathbf{1})$, and by virtue of the preceding inequalities,

$$0 \leqslant (1-x)\,h'(x)/h(x) \leqslant 1.$$

This shows that $h(x)$ is nondecreasing and $(1-x)\,h(x)$ is nonincreasing. Hence for $0 \leqslant x_1 < x_2 < 1$,

$$0 \leqslant h(x_2) - h(x_1) \leqslant h(x_1)(x_2-x_1)/(1-x_2).$$

Make the substitutions $x_1 = 1 - a_n$ and $x_2 = 1 - a_{n+1}$ obtaining

$$\begin{aligned} 0 &\leqslant h(1-a_{n+1}) - h(1-a_n) \\ &= \left(\frac{a_n}{a_{n+1}}\right)\left(\frac{a_n - a_{n+1}}{\mathbf{v}\cdot(\mathbf{F}(\mathbf{1}-a_n\mathbf{u})-\mathbf{F}_n(\mathbf{0}))}\right). \end{aligned}$$

As $n \to \infty$ $a_n/a_{n+1} \to 1$ and the second factor also approaches 1 by (2.1). This implies that $\limsup_{n\to\infty} h(1-a_n)/n \leqslant 1$, and so

$$\liminf_{n\to\infty} n \frac{\mathbf{v}\cdot(\mathbf{F}(\mathbf{1}-a_n\mathbf{u})-\mathbf{F}_n(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))} \geqslant 1.$$

The lemma then follows by invoking (2.1) once more. ▌

LEMMA 2. *Under the conditions of the theorem, for every integer* $k \geqslant 1$,

$$\lim_{n\to\infty} a_{nk}/a_n = m_k \tag{2.3}$$

exists and is positive.

Proof. We first remark that (1.5) implies that $[a_n\mathbf{Z}_n\cdot\mathbf{u} \mid \mathbf{Z}_n \neq \mathbf{0}, \mathbf{Z}_0 = \mathbf{e}_i]$ also converges in distribution by an argument along the same lines as in [2, Theorem 2]. In fact,

$$\lim_{n\to\infty} \Pr[a_n\mathbf{Z}_n\cdot\mathbf{u} \leqslant x \mid \mathbf{Z}_n \neq \mathbf{0}, \mathbf{Z}_0 = \mathbf{e}_i] = G(x\mathbf{v}\cdot\mathbf{w}; \mathbf{w}, i), \tag{2.4}$$

whenever x is a continuity point of $G(x; \mathbf{w}, i)$. We shall denote by $\phi(t)$ the Laplace–Stieltjes transform of the limiting distribution in (2.4) and by $\{\phi_n(t)\}$ the L.S. transforms of the approximating conditional distributions. Thus

$$\phi_n(t) = \frac{F_n^{(i)}(e^{-ta_n\mathbf{u}}) - F_n^{(i)}(\mathbf{0})}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))},$$

where $e^{\mathbf{s}}$ is interpreted as $(e^{s_1},..., e^{s_d})$. By [4, (3.3)] and vague convergence it follows that

$$\lim_{n\to\infty} 1 - \psi_n(t) = 1 - \phi(t), \qquad t > 0,$$

where

$$\psi_n(t) = \frac{\mathbf{v}\cdot(\mathbf{F}_n(e^{-ta_n\mathbf{u}}) - \mathbf{F}_n(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))}.$$

Our induction hypothesis is that (2.3) holds when k is the integer $j \geqslant 1$. We claim that the family $\{1 - \psi_n(t)\}$ is equicontinuous on $(0, \infty)$. To see this, suppose that $t > r$. Then

$$0 \leqslant 1 - \psi_n(t) - 1 + \psi_n(r) = \frac{\mathbf{v}\cdot(\mathbf{F}_n(e^{-ra_n\mathbf{u}}) - \mathbf{F}_n(e^{-ta_n\mathbf{u}}))}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))}.$$

Convexity (as in [3, Lemma 3.1]) implies this is less than

$$\frac{\mathbf{v}M^n(e^{-ra_n\mathbf{u}} - e^{-ta_n\mathbf{u}})}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))}$$

and the elementary inequality $0 \leqslant e^{-x} - e^{-y} \leqslant y - x$ valid for all $0 \leqslant x \leqslant y$ gives the upper bound

$$\frac{(t-r)\,\mathbf{v}M^n\mathbf{u}a_n}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))} \equiv t - r,$$

which certainly implies equicontinuity. By Ascoli's theorem we conclude that $1 - \psi_n(t) \to 1 - \psi(t)$ uniformly on compact t intervals (away from zero since convergence is only asserted for $t > 0$). By the induction hypothesis $\lim_{n\to\infty} a_{nj}/a_n = m_j > 0$. Hence if $t_n = sa_{nj}/a_n$ for any fixed $s > 0$, then

$$\lim_{n\to\infty} 1 - \psi_n(t_n) = 1 - \phi(sm_j). \tag{2.5}$$

Let

$$A_n^{(j+1)} = \frac{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_{n(j+1)}(\mathbf{0}))}{\mathbf{v}\cdot(\mathbf{1}-\mathbf{F}_n(\mathbf{0}))}.$$

We shall prove that $\lim_{n\to\infty} A_n^{(j+1)} = 1 - \phi(m_j)$ and this will therefore complete the proof of the lemma.

Given $\epsilon > 0$, for all n sufficiently large

$$\mathbf{1} - (1+\epsilon)\, a_{nj}\mathbf{u} \leqslant \mathbf{F}_{nj}(\mathbf{0}) \leqslant \mathbf{1} - (1-\epsilon)\, a_{nj}\mathbf{u}.$$

Since $x/(1 - e^{-x}) \to 1$ as $x \to 0$, given $\delta > 0$ for all $x > 0$ sufficiently small

$$1 - (1+\delta)x \leqslant e^{-x} \leqslant 1 - (1-\delta)x.$$

Since $a_{nj} \to 0$ as $n \to \infty$, given $\epsilon > 0$ and $\delta > 0$, then for all sufficiently large n

$$e^{-r_n a_n \mathbf{u}} \leqslant \mathbf{F}_{nj}(\mathbf{0}) \leqslant e^{-t_n a_n \mathbf{u}},$$

where $r_n = (1+\epsilon)\, a_{nj}/(1-\delta)\, a_n$ and $t_n = (1-\epsilon)\, a_{nj}/(1+\delta)\, a_n$. We find that

$$1 - \psi_n(r_n) \leqslant A_n^{(j+1)} \leqslant 1 - \psi_n(t_n)$$

and then by (2.5)

$$\begin{aligned} 1 - \phi(m_j(1-\epsilon)/(1+\delta)) &\leqslant \liminf_{n\to\infty} A_n^{(j+1)} \\ &\leqslant \limsup A_n^{(j+1)} \leqslant 1 - \phi(m_j(1+\epsilon)/(1-\delta)). \end{aligned}$$

Finally let $\epsilon \to 0$ and $\delta \to 0$. ∎

LEMMA 3 (Slack [8]). *Suppose that $c(t)$ is monotone tending to 0 as $t \to \infty$, $c(n+1)/c(n) \to 1$ as $n \to \infty$, and for every integer $k \geqslant 2$, $\lim_{n\to\infty} c(kn)/c(n)$ exists and is positive. Then*

$$\lim_{t\to\infty}(c(\lambda t)/c(t)) = \lambda^{\tau}$$

for some τ and all $\lambda > 0$.

COROLLARY. *Under the conditions of the theorem*

$$a_t = t^\tau L_1(t) \tag{2.6}$$

for some τ and L_1 a function slowly varying at infinity.

LEMMA 4. *Under the conditions of the theorem there is a constant* $\alpha \in (0, 1]$ *such that*

$$\lim_{n\to\infty} n \frac{\mathbf{v} \cdot (\mathbf{F}_{n+1}(0) - \mathbf{F}_n(0))}{\mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(0))} = \frac{1}{\alpha}. \tag{2.7}$$

Proof. From (1.2) we obtain $\mathbf{v} \cdot (\mathbf{F}(\mathbf{s}) - \mathbf{s}) = \mathbf{v}E(\mathbf{s})(\mathbf{1} - \mathbf{s})$ where the right-hand side is nonincreasing. Mimicking the corresponding proof in [8],

$$\lim_{n\to\infty} n \frac{(a_n - a_{n+1})}{a_n} = -\tau.$$

But by Lemma 1 $-\tau \geqslant 1$. Hence defining $\alpha = -1/\tau$, we have the bounds $0 < \alpha \leqslant 1$. The lemma then obtains and we may write (2.6) as

$$a_t = t^{-(1/\alpha)} L_1(t). \quad \blacksquare \tag{2.8}$$

LEMMA 5. *For $0 < x \leqslant a_1$ define $k = k(x)$ by the inequalities*

$$a_{k+1} < x \leqslant a_k. \tag{2.9}$$

Then $k(x)$ is regularly varying at 0 with index $-\alpha$.

Proof. From (2.8),

$$1/t \sim (1/a_t)^{-\alpha} (1/L_1(t))^\alpha, \qquad t \to \infty.$$

Using [5, Lemma 3] we then deduce that

$$1/t \sim (1/a_t)^{-\alpha} L_2(1/a_t), \qquad t \to \infty, \tag{2.10}$$

where L_2 is some other function slowly varying at ∞. By (2.9), $a_{k(x)}/x \to 1$ as $x \to 0$ and thus by the uniform convergence property of slowly varying functions [6], $L_2(1/a_{k(x)}) \sim L_2(1/x)$, $x \to 0$. Substituting $t = k(x)$ into (2.10) we obtain finally

$$1/k(x) \sim (1/x)^{-\alpha} L_2(1/x), \qquad x \to 0,$$

which is precisely the assertion of the lemma. $\blacksquare$

3. Proof of Theorem

(a) From (2.9)

$$1 - a_k\mathbf{u} \leqslant 1 - x\mathbf{u} < 1 - a_{k+1}\mathbf{u}.$$

Invoking the monotonicity of $\mathbf{v} \cdot (\mathbf{F}(\mathbf{s}) - \mathbf{s})$,

$$\begin{aligned}\frac{\mathbf{v} \cdot (\mathbf{F}(\mathbf{1} - a_{k+1}\mathbf{u}) - (\mathbf{1} - a_{k+1}\mathbf{u}))}{a_k} &\leqslant \frac{\mathbf{v} \cdot (\mathbf{F}(\mathbf{1} - x\mathbf{u}) - (\mathbf{1} - x\mathbf{u}))}{x} \\ &\equiv \Lambda(x) \\ &\leqslant \frac{\mathbf{v} \cdot (\mathbf{F}(\mathbf{1} - a_k\mathbf{u}) - (\mathbf{1} - a_k\mathbf{u}))}{a_{k+1}}.\end{aligned}$$

If we multiply these inequalities by $k(x)$, apply (2.1) and Lemma 4, and then let $x \to 0$, we will have

$$\lim_{x \to 0} k(x)\, \Lambda(x) = 1/\alpha.$$

Finally from Lemma 5 we conclude (1.3) holds with the choice $L(x) = \alpha^{-1}L_2(1/x)$.

(b) This follows trivially from [2, Theorem 2], from which we can also find the simple relationship connecting $H(\mathbf{s})$ and $G(x; \mathbf{w}, i)$, namely,

$$H(\mathbf{s}) = G(\min_{1 \leqslant i \leqslant d} (s_i/v_i)\mathbf{v} \cdot \mathbf{w}; \mathbf{w}, i). \quad \blacksquare$$

4. Concluding Remarks

(a) Suppose for some constant $c > 0$,

$$\mathbf{v} \cdot (\mathbf{1} - \mathbf{F}_n(\mathbf{0})) \sim cn\mathbf{v} \cdot (\mathbf{F}_{n+1}(\mathbf{0}) - \mathbf{F}_n(\mathbf{0})).$$

Then (1.3) holds and $c = \alpha$. The details follow the one-dimensional argument [8].

(b) If $\mathbf{w}$ is such that $\mathbf{v} \cdot \mathbf{w} = 0$ then one can show, under (1.3), that $[a_n\mathbf{Z}_n \cdot \mathbf{w} \mid \mathbf{Z}_n \neq \mathbf{0}] \to 0$ in distribution. When all second moments are finite, Athreya and Ney [1] have obtained the correct normalization for

$$[\mathbf{Z}_n \cdot \mathbf{w} \mid \mathbf{Z}_n \neq \mathbf{0}].$$

It is thus of some interest to determine the analogous results in the present context. We make the obvious conjecture that stable laws will enter the picture.

(c) The function $G(x; \mathbf{w}, i)$ appearing in the proof of the theorem is the limiting conditional distribution of $a_n\mathbf{Z}_n$ projected on $\mathbf{w}$. In [2, Theorem 1] the natural direction for projection was $\mathbf{u}$ and the resulting limiting distribution was denoted by $G(x)$. The explicit relationship between these functions is

$$G(x\mathbf{v} \cdot \mathbf{w}; \mathbf{w}, i) = G(x).$$

Using this and [2, Theorem 2] we find that $H(\mathbf{s})$ is the distribution function of a random vector $W\mathbf{v}$ concentrated on the ray $c\mathbf{v}$, $c > 0$. W is a random variable whose Laplace–Stieltjes transform is of the form $1 - t(1 + t^{\alpha})^{-1/\alpha}$.

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